

Mersenne conjectures

In mathematics, the **Mersenne conjectures** concern the characterization of prime numbers of a form called Mersenne primes, meaning prime numbers that are a power of two minus one.

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Original Mersenne conjecture

The original, called **Mersenne's conjecture**, was a statement by Marin Mersenne in his *Cogitata Physico-Mathematica* (1644; see e.g. Dickson 1919) that the numbers $2^n - 1$ were prime for $n = 2, 3, 5, 7, 13, 17, 19, 31, 67, 127$ and 257, and were composite for all other positive integers $n \leq 257$. Due to the size of these numbers, Mersenne did not and could not test all of them, nor could his peers in the 17th century. It was eventually determined, after three centuries and the availability of new techniques such as the Lucas–Lehmer test, that Mersenne's conjecture contained five errors, namely two are composite ($n = 67, 257$) and three omitted primes ($n = 61, 89, 107$). The correct list is: $n = 2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107$ and 127.

While Mersenne's original conjecture is false, it may have led to the **New Mersenne conjecture**.

New Mersenne conjecture

The **New Mersenne conjecture** or **Bateman, Selfridge and Wagstaff conjecture** (Bateman et al. 1989) states that for any odd natural number p , if any two of the following conditions hold, then so does the third:

- $p = 2^k \pm 1$ or $p = 4^k \pm 3$ for some natural number k . (OEIS: A122834)
- $2^p - 1$ is prime (a Mersenne prime). (OEIS: A000043)
- $(2^p + 1) / 3$ is prime (a Wagstaff prime). (OEIS: A000978)

If p is an odd composite number, then $2^p - 1$ and $(2^p + 1)/3$ are both composite. Therefore it is only necessary to test primes to verify the truth of the conjecture.

Currently, the known numbers for which all three conditions hold are: 3, 5, 7, 13, 17, 19, 31, 61, 127 (sequence A107360 in the OEIS). It is also a conjecture that no number which is greater than 127 satisfies all three conditions. As of February 2020 all the Mersenne primes up to $2^{43112609}-1$ are known, and for none of these does the third condition hold except for the ones just mentioned.^[1]

Primes which satisfy at least one condition are

2, 3, 5, 7, 11, 13, 17, 19, 23, 31, 43, 61, 67, 79, 89, 101, 107, 127, 167, 191, 199, 257, 313, 347, 521, 607, 701, 1021, 1279, 1709, 2203, 2281, 2617, 3217, 3539, 4093, 4099, 4253, 4423, 5807, 8191, 9689, 9941, ... (sequence [A120334](#) in the [OEIS](#))

Note that the two primes for which the original Mersenne conjecture is false (67 and 257) satisfy the first condition of the new conjecture ($67=2^6+3$, $257=2^8+1$), but not the other two. 89 and 107, which were missed by Mersenne, satisfy the second condition but not the other two. Mersenne may have thought that $2^p - 1$ is prime only if $p = 2^k \pm 1$ or $p = 4^k \pm 3$ for some natural number k , but if he thought it was "if and only if" he would have included 61.

Status of new Mersenne conjecture for the first 100 primes

2	3	5	7	11	13	17	19	23	29
31	37	41	43	47	53	59	61	67	71
73	79	83	89	97	101	103	107	109	113
127	131	137	139	149	151	157	163	167	173
179	181	191	193	197	199	211	223	227	229
233	239	241	251	257	263	269	271	277	281
283	293	307	311	313	317	331	337	347	349
353	359	367	373	379	383	389	397	401	409
419	421	431	433	439	443	449	457	461	463
467	479	487	491	499	503	509	521	523	541

Red: p is of the form $2^n \pm 1$ or $4^n \pm 3$	Cyan background: $2^p - 1$ is prime	<i>Italics: $(2^p + 1)/3$ is prime</i>	Bold: p satisfies at least one condition
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The New Mersenne conjecture can be thought of as an attempt to salvage the centuries-old Mersenne's conjecture, which is false. However, according to [Robert D. Silverman](http://www.mersenneforum.org/showpost.php?p=53533&postcount=3) (<http://www.mersenneforum.org/showpost.php?p=53533&postcount=3>), John Selfridge agreed that the New Mersenne conjecture is "obviously true" as it was chosen to fit the known data and counter-examples beyond those cases are exceedingly unlikely. It may be regarded more as a curious observation than as an open question in need of proving.

Renaud Lifchitz has shown that the NMC is true for all integers less than or equal to 30,402,456^[2] by systematically testing all primes for which it is already known that one of the conditions holds. [His website](http://www.primenumbers.net/rl/nmc/) (<http://www.primenumbers.net/rl/nmc/>) documents the verification of results up to this number. Another, currently more up-to-date status page on the NMC is [The New Mersenne Prime conjecture](http://primes.utm.edu/mersenne/NewMersenneConjecture.html). (<http://primes.utm.edu/mersenne/NewMersenneConjecture.html>)

Lenstra–Pomerance–Wagstaff conjecture

[Lenstra](#), [Pomerance](#), and [Wagstaff](#) have conjectured that there is an infinite number of [Mersenne primes](#), and, more precisely, that the number of Mersenne primes less than x is asymptotically approximated by

$$e^\gamma \cdot \log_2 \log_2(x),^{[3]}$$

where γ is the [Euler–Mascheroni constant](#). In other words, the number of Mersenne primes with exponent p less than y is asymptotically

$$e^\gamma \cdot \log_2(y).^{[3]}$$

This means that there should on average be about $e^\gamma \cdot \log_2(10) \approx 5.92$ primes p of a given number of decimal digits such that M_p is prime. The conjecture is fairly accurate for the first 40 Mersenne primes, but between $2^{20,000,000}$ and $2^{85,000,000}$ there are at least 12,^[4] rather than the expected number which is around 3.7.

More generally, the number of primes $p \leq y$ such that $\frac{a^p - b^p}{a - b}$ is prime (where a, b are coprime integers, $a > 1$, $-a < b < a$, a and b are not both perfect r -th powers for any natural number $r > 1$, and $-4ab$ is not a perfect fourth power) is asymptotically

$$(e^\gamma + m \cdot \log_e(2)) \cdot \log_a(y).$$

where m is the largest nonnegative integer such that a and $-b$ are both perfect 2^m -th powers. The case of Mersenne primes is one case of $(a, b) = (2, 1)$.

See also

- Gillies' conjecture on the distribution of numbers of prime factors of Mersenne numbers
- Lucas–Lehmer primality test
- Lucas primality test
- Catalan's Mersenne conjecture
- Mersenne's laws

References

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External links

- The Prime Pages. Mersenne's conjecture. (http://primes.utm.edu/glossary/page.php?sort=MersennesConjecture)

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